Enumerating the edge-colourings and total colourings of a regular graph

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2012 Workshop on Graph Theory and Combinatorics
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Kaohsiung, Taiwan.

Outline

Introduction

Vertex Coloring

Definition (k-vertex coloring)

A *k-vertex coloring* of a (non oriented) graph $G = (V, E)$ is a function $c : V \to \{1, \ldots, k\}$ such that $uv \in E$ implies $c(u) \neq c(v)$.

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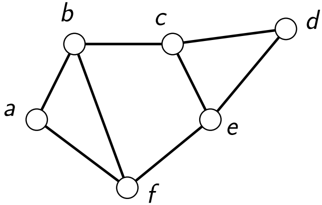

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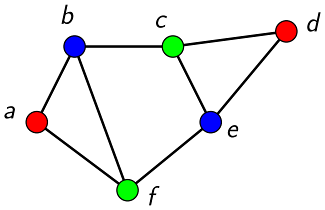

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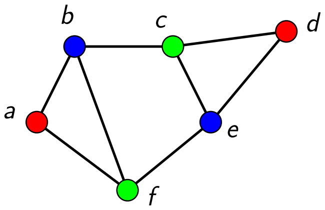

Definition (chromatic number)

$\chi(G)$ is the minimum k such that G admits a k-vertex coloring.

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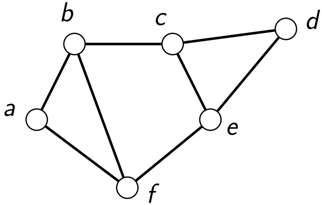

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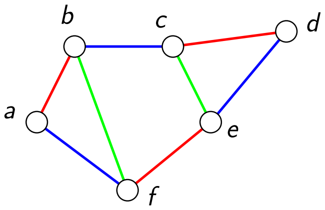

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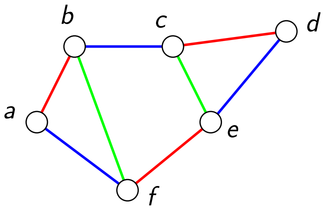

Definition (chromatic index)

$\chi'(G)$ is the minimum k such that G admits a k-edge coloring.

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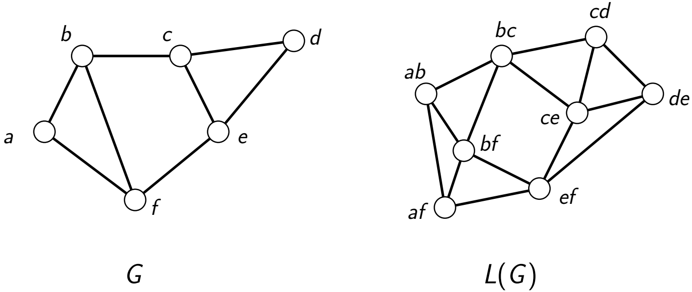
- The *line graph* $L(G)$ of G:
  - The vertex set of $L(G)$ is the edge set of G
  - Two edges e and f of G are adjacent in $L(G)$ if there are incident in G



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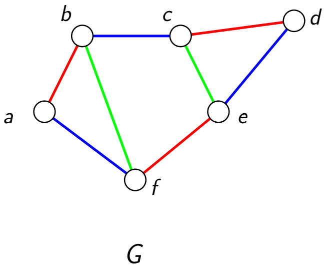

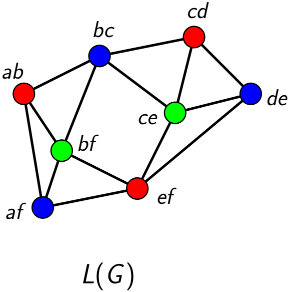

- $\chi'(G) = \chi(L(G))$

Total Coloring

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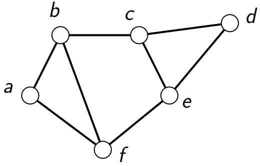

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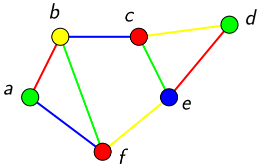

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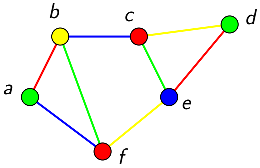

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Famous bounds

Let $\Delta(G)$ denotes the maximum degree of G.

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- Vizing's Theorem (1964): $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$
- Total coloring conjecture (Behzad, Vizing, $\sim$1964): $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$

Algorithmic problems

k-VERTEX/EDGE/TOTAL COLORING:

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- **exact (exponential) algorithms**: find an algorithm with running time $O^\star(c^n)$ $(= O(P(n)c^n))$

Examples of exact algorithms

ENUM-3-EDGE COLORING:

- *Input:* a graph $G = (V, E)$ with $\Delta(G) \leq 3$.
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Usual methods to improve: **branching algorithms** and dynamic programming.

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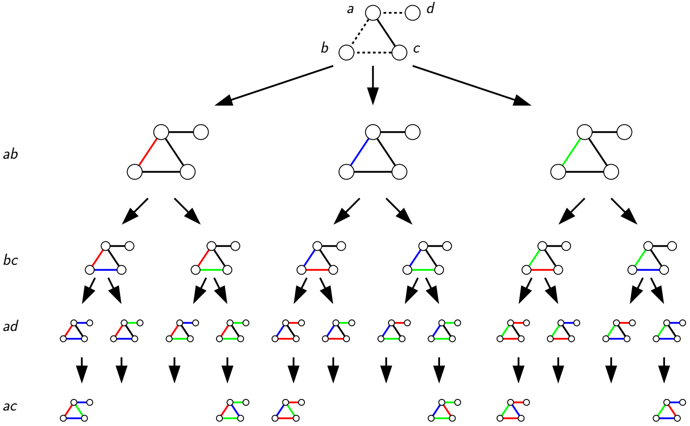
ENUM-3-EDGE COLORING: a 'naive' branching algorithms



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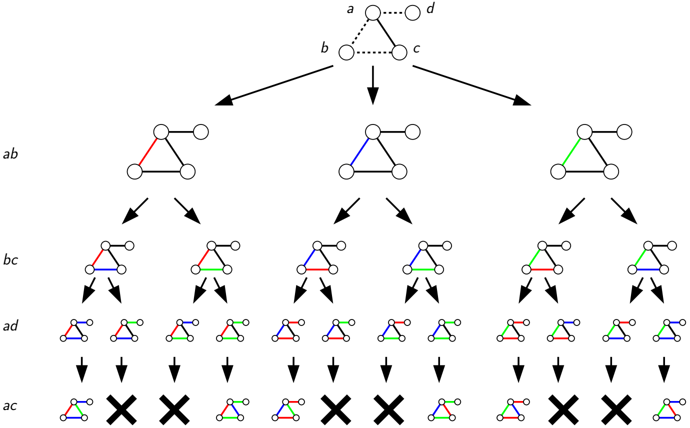

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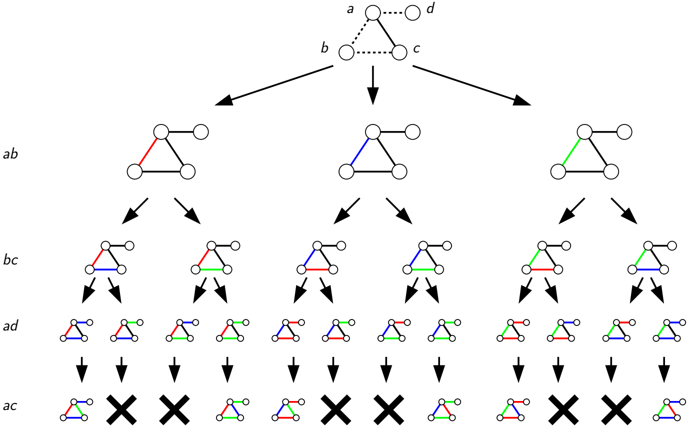

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Polynomial space.

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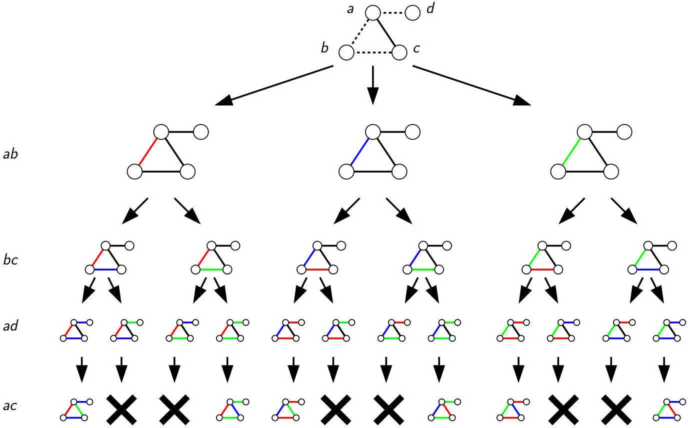

Time analysis: $O^{\star}(2^m) = O^{\star}(2^{\frac{3}{2}n}) = O^{\star}(2.8284^n)$ min. nb of leaves

Polynomial space. min. height

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In the same environment:

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Here: $\frac{\log 3^{\frac{3}{2}}}{\log 2^{\frac{3}{2}}} = 1.58$

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- ENUM-k-EDGE COLORING in $O^\star((k-1!)^{\frac{n}{2}})$ and a sharp example (multi-graph).
- More precise bound on the number of 3-edge coloring of simple graph

Enumerating the 3-edge colorings

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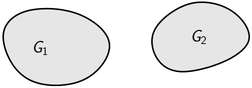

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Enum-3-edge coloring

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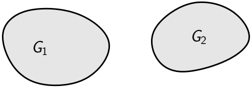
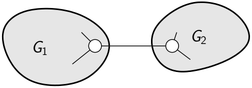


$$c_3(G) = c_3(G_1) \cdot c_3(G_2)$$

We can assume that G is 2-(vertex) connected.



$$c_3(G) = \frac{1}{3} \cdot c_3(G_1) \cdot c_3(G_2)$$

Number of 3-edge colorings of 3-regular graphs

Lemma

Let C_n be the cycle of length n.

$$c_3(C_n) = \begin{cases} 2^n + 2, & \text{if } n \text{ is even,} \\ 2^n - 2, & \text{if } n \text{ is odd.} \end{cases}$$

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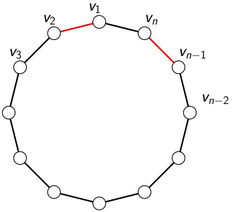
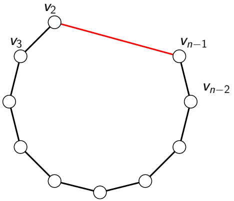
Proof: By induction on n. True for $n = 2$ and $n = 3$.





$$c_3(C_n) = 2.c_3(C_{n-2}) +$$

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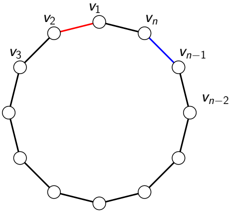

$$c_3(C_n) = 2.c_3(C_{n-2}) + c_3(C_{n-1}) \qquad \square$$

Number of 3-edge colorings of 3-regular graphs

We denote by n_i the number of vertices of degree i in G.

Theorem

Let G be a 2-connected subcubic graph. Then $c_3(G) \leq 3 \cdot 2^{n-\frac{n_3}{2}}$.

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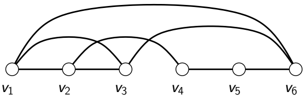
for all $1 < i < n$, $\quad d(v_i)_{\{v_1,\ldots,v_{i-1}\}} \geq 1$ and $d(v_i)_{\{v_{i+1},\ldots,v_n\}} \geq 1$.

(always exists: Lempel et al.,1967)



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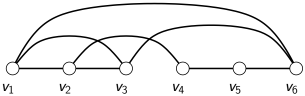

Orient from left to right.
$A^+ = \{v \ : \ d^+(v) = 2 \text{ and } d^-(v) = 1\}$
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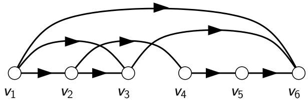

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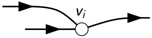
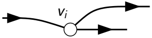
- $c_1 = 6$
- if $v_i \in A^-$ 

$c_i \leq c_{i-1}$
- if $v_i \in A^+$ 

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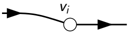
- if $d^+(v_i) = d^-(v_i) = 1$ 

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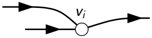
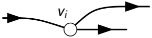
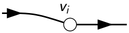
- if $v_i \in A^-$  $c_i \leq c_{i-1}$
- if $v_i \in A^+$  $c_i \leq 2c_{i-1}$
- if $d^+(v_i) = d^-(v_i) = 1$  $c_i \leq 2c_{i-1}$

$c_3(G) \leq 6 \cdot 2^{(n-2)-(n_3-2)/2} = 3 \cdot 2^{n-\frac{n_3}{2}}$ □

Number of 3-edge colorings of 3-regular graphs

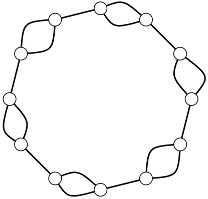
Theorem

Let G be a 2-connected subcubic graph. Then $c_3(G) \leq 3 \cdot 2^{n-\frac{n_3}{2}}$.

Corollary

Let G be a cubic graph, then $c_3(G) \leq 3 \cdot 2^{\frac{n}{2}}$.

This is sharp for multi-graphs:



ENUM-3-EDGE COLORING:

Corollary (Solving ENUM-3-EDGE COLORING:)

There exists a branching algorithm with running time $O^\star(2^{\frac{n}{2}}) = O^\star(1.4143^n)$ *and polynomial space to solve* ENUM-3-EDGE COLORING.

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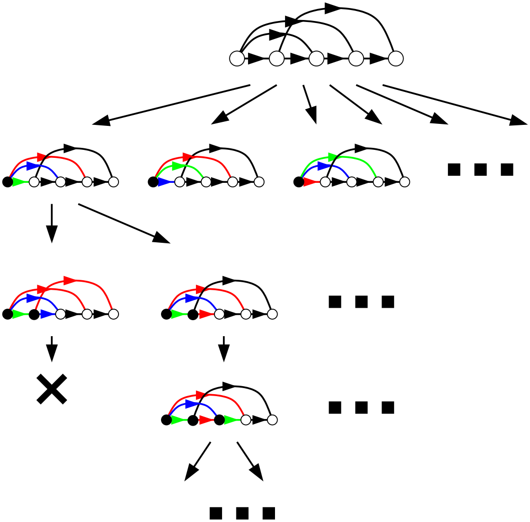
Proof:

- *If* $\Delta \geq 4$ answer 0.
- If the graph is not connected enough, divide the instance.
- Otherwise, run a branching algorithm.

Enum-3-edge coloring:



Extensions

1. Introduction
   - Colorings
   - Algorithmic problems
   - Our results
2. Enumerating the 3-edge colorings of a cubic graph
   - The 3-edge colorings of a 3-regular graph
   - Turning the proof into algorithm
3. Extensions: k-edge colorings and the total colorings
   - k-edge colorings
   - A more precise bounds for the 3-edge coloring
   - Total coloring
4. Conclusion

Number of k-edge colorings of k-regular graphs

We denote by c_k the number of k-edge colorings of G.

Theorem

Let G be a k-regular graph. Then $c_k(G) \leq k \cdot (k-1!)^{\frac{n}{2}}$.

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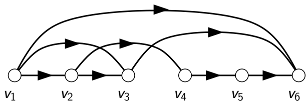

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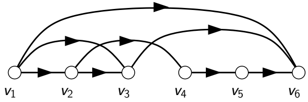
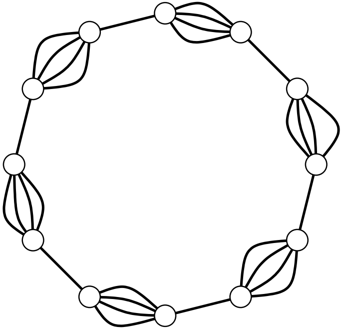


$A_i = \{v \ : \ d^+(v) = i\}$ and $a_i = |A_i|$ for $i = 1, \ldots, k-1$

Then, $c_k(G) \leq \prod_{i=1}^{n-1} d^+(v_i)! \leq k! \prod_{i=1}^{k-1} (i!)^{a_i}$
under $\sum_{i=1}^{k-1} a_i = n-2$ and $\sum_{i=1}^{k-1} i.a_i = k(n-2)/2$ $\square$

Number of k-edge colorings of k-regular graphs

Also sharp for multi-graphs:



Enum-k-edge coloring:

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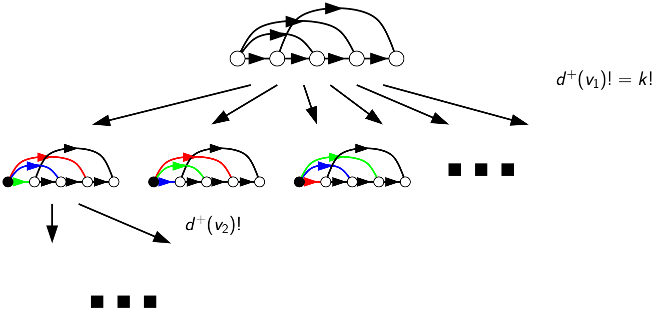
Proof:



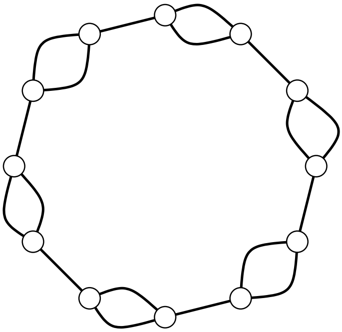
At most $\prod_{i=1}^{n-1} d^+(v_i)! = O^\star((k-1!)^{\frac{n}{2}})$ leaves

Number of 3-edge colorings for simple graphs

Corollary

Let G be a cubic graph, then $c_3(G) \leq 3 \cdot 2^{\frac{n}{2}}$.

The sharp example for the number of 3-edge coloring of cubic graphs:



Number of 3-edge colorings for simple graphs

Can we improve a lot?

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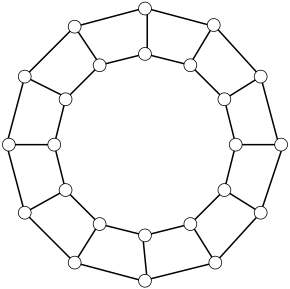

Lemma (c_3 of the ladder graph)

$$c_3(H_n) = \begin{cases} 2^{n/2} + 8, & \textit{if } n/2 \textit{ is even,} \\ 2^{n/2} - 2, & \textit{if } n/2 \textit{ is odd.} \end{cases}$$

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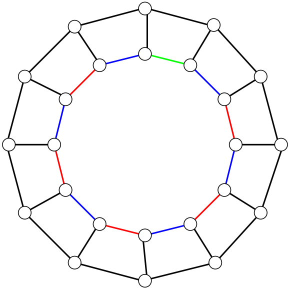

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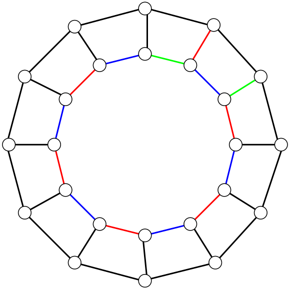

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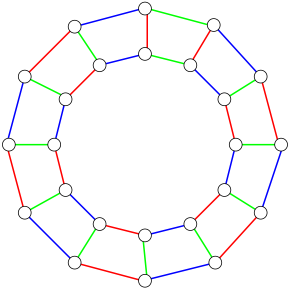

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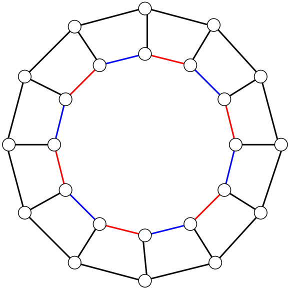

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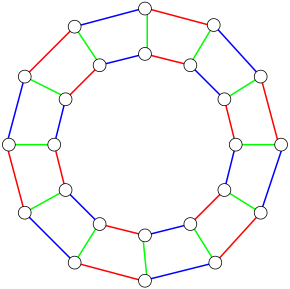

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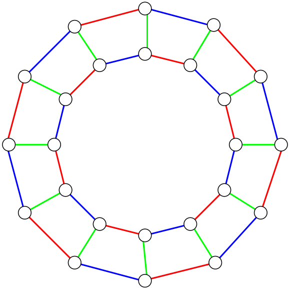

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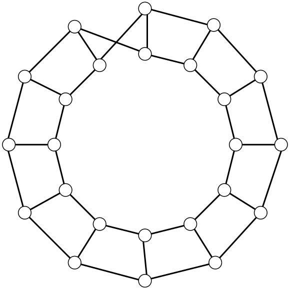

Lemma (c_3 of the Mœbius ladder graph)

$$c_3(M_n) = \begin{cases} 2^{n/2} + 2, & \textit{if } n/2 \textit{ is even,} \\ 2^{n/2} + 4, & \textit{if } n/2 \textit{ is odd.} \end{cases}$$

Number of 3-edge colorings for simple graphs

So, for simple cubic graphs, we have:

$$2^{\frac{n}{2}} < \max\{c_3(G) : G \text{ cubic graph on } n \text{ vertices }\} \leq 3.2^{\frac{n}{2}}$$

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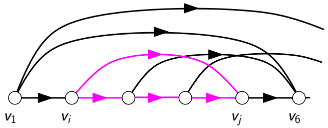

We have $\frac{1}{3} \cdot c_3(C_{j-i+1})/2^{j-i} \cdot 3.2^{\frac{n}{2}}$ colorings.

4-total colorings

Let $c_4^T(G)$ be the number of 4-total colorings of G.

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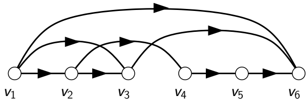

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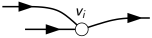
- if $v_i \in A^-$  $c_i \leq 2c_{i-1}$

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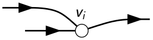
- if $v_i \in A^-$ 

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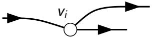
- if $v_i \in A^+$ 

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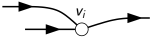
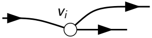
- if $v_i \in A^-$  $c_i \leq 2c_{i-1}$
- if $v_i \in A^+$  $c_i \leq 4c_{i-1}$

Hence:

$c_4^T(G) \leq 24 \cdot 2^{\frac{(n-2)}{2}} \cdot 4^{\frac{(n-2)}{2}} = 3 \cdot 2^{\frac{3n}{2}}$ □

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Corollary (Solving Enum-3-edge coloring:)

There exists a branching algorithm with running time $O^\star(2^{\frac{3n}{2}}) = O^\star(2.8285^n)$ *and polynomial space to solve* Enum-4-total coloring.

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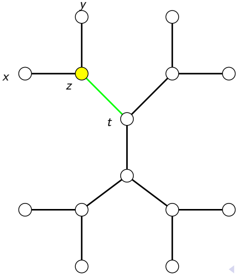

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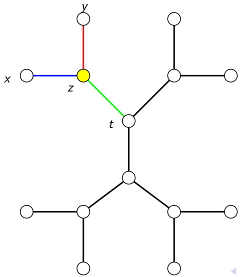

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Let T_n be a binary tree on n vertices + parallel edges between twin leaves, $c_4^T(T_n) = \frac{3}{\sqrt{2}} \cdot 2^{\frac{5n}{4}}$.

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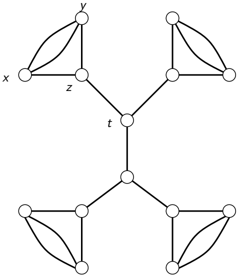

$n = p - 2 + p + 2p$

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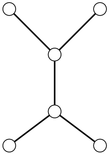

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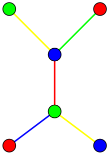

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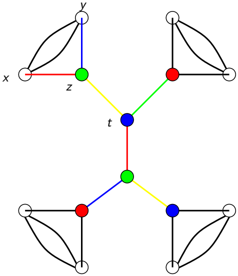

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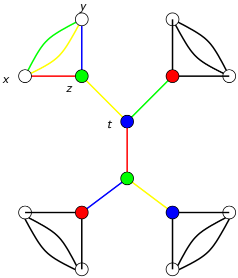

$$c_4^T(T') = 3 \cdot 2^{\frac{3(2p-2)}{2}} \cdot 4^p = 3 \cdot 2^{5p-3} = 3 \cdot 2^{\frac{5n}{4} - \frac{1}{2}}$$

Conclusion

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- Minimum vertex coloring in $O^\star(2^n)$ (A. Bjöklund, T. Husfeld and M. Koivisto, 2006)

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  Under ETH, there is no $O^\star(2^{o(n)})$ algorithm for 3-VERTEX COLORING (D. Lokshtanov, D. Marx, S. Saurabh, 2012), several scenarios...

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  Is it possible to find the algorithm with best running time for ENUM-4-TOTAL COLORING.

Thank you for your attention !